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ON THE EXPRESSION OF CERTAIN MINORS OF THE LTH COMPOUND OF A DETERMINANT AS A FUNCTION OF THE ELEMENTS OF A SINGLE LINE OF THE MTH COMPOUND.

By W. H. Metzler.

In the Messenger of Mathematics for December, 1905, Dr. Thos. Muir called attention to a relation between two determinants of orders n and $(n-1)$, which arose in connection with a problem of elimination. In this paper a slightly more general relation is considered from a somewhat different angle, adding sufficient novelty for further interest.

If

$$\Delta \equiv \left(\begin{vmatrix} (l_1) \\ (n \mid \underset{\alpha}{m} \mid \underset{1}{l}) \end{vmatrix} \begin{vmatrix} (l_2) \\ (n \mid \underset{\alpha}{m} \mid \underset{2}{l}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda) \\ (n \mid \underset{\alpha}{m} \mid \underset{\lambda}{l}) \end{vmatrix} \right),$$

and

$$\Delta' \equiv \left(\begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{1}{l}) \\ (n \mid \underset{\alpha}{\overline{m}} \mid \underset{1}{l}) \end{vmatrix} \begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{2}{l}) \\ (n \mid \underset{\alpha}{m} \mid \underset{2}{l}) \end{vmatrix} \cdots \begin{vmatrix} (n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m} \mid \underset{\lambda}{l}) \end{vmatrix} \right),$$

where (l_1), (l_2), $\cdots$, (l_λ) are λ sets of l numbers each taken from the numbers $1, 2, 3 \cdots, n$; and where no two sets have any numbers in common; then the product

$$\Delta \cdot \Delta' = \Delta'' \equiv \left(\begin{vmatrix} (l_1)(n \mid \underset{\beta}{\overline{m}} \mid \underset{1}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \begin{vmatrix} (l_2)(n \mid \underset{\beta}{\overline{m}} \mid \underset{2}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda)(n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \right). \qquad (1)$$

It may be observed that while Δ is a minor of the lth compound of a given determinant, and Δ' is a minor of the $(m-l)$th compound of this same determinant, Δ'' is not a minor of the mth compound of this determinant, but a function of the elements of a single column of the mth compound.

It should also be observed that the row numbers of Δ'' are the row numbers of Δ combined with the corresponding row numbers of Δ'.

Suppose now that $m = k \cdot l + h$, where $h < l$ and therefore k is the greatest integer in m/l.

Let

$$(l_1)(l_2)(l_3) \cdots (l_k)(h) = (n \mid \underset{\beta}{m}),$$

then obviously

$$\Delta'' = \Delta''' \begin{vmatrix} (n \mid \underset{\beta}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{k}, \tag{2}$$

where

$$\Delta''' \equiv \left(\begin{vmatrix} (l_{k+1})(n \mid \underset{\beta}{\overline{m}} \mid \underset{k+1}{l}\,) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (l_\lambda)(n \mid \underset{\beta}{\overline{m}} \mid \underset{\lambda}{l}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \right).$$

Again, since

$$\Delta' = \begin{vmatrix} (n \mid \underset{\beta}{m}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{(m-1)_l} \equiv M^{(m-1)_l},$$

say, we have

$$\Delta \cdot M^{(m-1)_l - k} = \Delta'''. \tag{3}$$

This may be looked upon as a relation giving the value of Δ in terms of Δ''', or giving the value of Δ''' in terms of Δ.

If in (3) $l = m - 1$, then $k = 1$, and $h = 1$, and we have

$$\Delta = \Delta''', \tag{4}$$

a result given by Muir.*

If

$$\begin{aligned} (l_1) &\equiv (\alpha_1\alpha_2 \cdots \alpha_{m-1}) \equiv (n \mid \underset{\beta}{m} \mid \underset{1}{l}), \\ (l_2) &\equiv (\alpha_2\alpha_3 \cdots \alpha_m), \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ (l_\lambda) &\equiv (\alpha_m\alpha_{m+1} \cdots \alpha_{2m-1}), \end{aligned}$$

then

$$\Delta = \Delta''' = \begin{vmatrix} (\alpha_1\alpha_2 \cdots \alpha_m) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdot \begin{vmatrix} (\alpha_2\alpha_3 \cdots \alpha_{m+1}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix} \cdots \begin{vmatrix} (\alpha_m\alpha_{m+1} \cdots \alpha_{2m-1}) \\ (n \mid \underset{\alpha}{m}) \end{vmatrix}^{*}. \tag{5}$$

As an example of (3) let $m = 4$, $l = 2$, $(n \mid \underset{\beta}{m}) \equiv (9\tau\sigma\rho)$, and $(n \mid \underset{\alpha}{m}) \equiv (1234)$ and we have

$$\left(\begin{vmatrix} 12 \\ 12 \end{vmatrix} \begin{vmatrix} 34 \\ 13 \end{vmatrix} \begin{vmatrix} 56 \\ 14 \end{vmatrix} \begin{vmatrix} 78 \\ 23 \end{vmatrix} \begin{vmatrix} 9\tau \\ 24 \end{vmatrix} \begin{vmatrix} \sigma\rho \\ 34 \end{vmatrix} \right) \times \begin{vmatrix} 9\tau\sigma\rho \\ 1234 \end{vmatrix} = \left(\begin{vmatrix} 12\tau\rho \\ 1234 \end{vmatrix} \begin{vmatrix} 34\tau\sigma \\ 1234 \end{vmatrix} \begin{vmatrix} 569\rho \\ 1234 \end{vmatrix} \begin{vmatrix} 789\sigma \\ 1234 \end{vmatrix} \right).$$

Here $k = 2$, and the row numbers of Δ''' are formed by adding to the row numbers 12, 34, 56, 78, of Δ, the row numbers $\tau\rho$, $\tau\sigma$, 9ρ, 9σ, which are the combinations of $9\tau\sigma\rho$ taken two at a time except those two which appear in Δ, viz. 9τ and $\sigma\rho$.

If in this example we had taken $(n \mid \underset{\beta}{m}) \equiv (1234)$ we would have

$$\left(\begin{vmatrix} 12 \\ 12 \end{vmatrix} \begin{vmatrix} 34 \\ 13 \end{vmatrix} \begin{vmatrix} 56 \\ 14 \end{vmatrix} \begin{vmatrix} 78 \\ 23 \end{vmatrix} \begin{vmatrix} 9\tau \\ 24 \end{vmatrix} \begin{vmatrix} \sigma\rho \\ 34 \end{vmatrix} \right) \begin{vmatrix} 1234 \\ 1234 \end{vmatrix} = \left(\begin{vmatrix} 1356 \\ 1234 \end{vmatrix} \begin{vmatrix} 1478 \\ 1234 \end{vmatrix} \begin{vmatrix} 239\tau \\ 1234 \end{vmatrix} \begin{vmatrix} 24\sigma\rho \\ 1234 \end{vmatrix} \right).$$

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* Ibid. Cf. author's paper on Compound Determinants, Amer. Jour. Math., vol. 20, No. 3.

If each element in the first column of Δ is the sum of two terms we would have

$$\left(\begin{vmatrix}12\\12\end{vmatrix}+\begin{vmatrix}pq\\12\end{vmatrix}\begin{vmatrix}34\\13\end{vmatrix}\begin{vmatrix}56\\14\end{vmatrix}\begin{vmatrix}78\\23\end{vmatrix}\begin{vmatrix}9\tau\\24\end{vmatrix}\begin{vmatrix}\sigma\rho\\34\end{vmatrix}\right)\times\begin{vmatrix}9\tau\sigma\rho\\1234\end{vmatrix}$$

$$=\left(\begin{vmatrix}12\tau\rho\\1234\end{vmatrix}+\begin{vmatrix}pq\tau\rho\\1234\end{vmatrix}\begin{vmatrix}34\tau\sigma\\1234\end{vmatrix}\begin{vmatrix}569\rho\\1234\end{vmatrix}\begin{vmatrix}789\sigma\\1234\end{vmatrix}\right).$$

In general, if

$$S_k=\sum_{i=1}^{h_k}\begin{vmatrix}(l_k{}^i)\\(n\mid \underset{a}{m}\mid \underset{k}{l})\end{vmatrix},$$

and

$$\Delta\equiv(S_1S_2\cdots S_\lambda),$$

then

$$\Delta\cdot\Delta'=\Delta''=(S_1'S_2'\cdots S_\lambda'),\tag{6}$$

where

$$S_k'=\sum_{i=1}^{h_k}\begin{vmatrix}(l_k{}^i)(n\mid\underset{\beta}{\overline{m}}\mid\underset{k}{l})\\(n\mid\underset{a}{m})\end{vmatrix}.$$

A few further examples will illustrate with what ease the value of such determinants may be written down.

Taking $l=m-1$, we have

$$\begin{Vmatrix}
\begin{vmatrix}123\\123\end{vmatrix}\begin{vmatrix}126\\123\end{vmatrix}-\begin{vmatrix}135\\123\end{vmatrix}+\begin{vmatrix}234\\123\end{vmatrix}\begin{vmatrix}156\\123\end{vmatrix}-\begin{vmatrix}246\\123\end{vmatrix}+\begin{vmatrix}345\\123\end{vmatrix}\begin{vmatrix}456\\123\end{vmatrix}\\
\begin{vmatrix}123\\124\end{vmatrix}\begin{vmatrix}126\\124\end{vmatrix}-\begin{vmatrix}135\\124\end{vmatrix}+\begin{vmatrix}234\\124\end{vmatrix}\begin{vmatrix}156\\124\end{vmatrix}-\begin{vmatrix}246\\124\end{vmatrix}+\begin{vmatrix}345\\124\end{vmatrix}\begin{vmatrix}456\\124\end{vmatrix}\\
\begin{vmatrix}123\\134\end{vmatrix}\begin{vmatrix}126\\134\end{vmatrix}-\begin{vmatrix}135\\134\end{vmatrix}+\begin{vmatrix}234\\134\end{vmatrix}\begin{vmatrix}156\\134\end{vmatrix}-\begin{vmatrix}246\\134\end{vmatrix}+\begin{vmatrix}345\\134\end{vmatrix}\begin{vmatrix}456\\134\end{vmatrix}\\
\begin{vmatrix}123\\234\end{vmatrix}\begin{vmatrix}126\\234\end{vmatrix}-\begin{vmatrix}135\\234\end{vmatrix}+\begin{vmatrix}234\\234\end{vmatrix}\begin{vmatrix}156\\234\end{vmatrix}-\begin{vmatrix}246\\234\end{vmatrix}+\begin{vmatrix}345\\234\end{vmatrix}\begin{vmatrix}456\\234\end{vmatrix}
\end{Vmatrix}$$

$$=\begin{Vmatrix}
\begin{vmatrix}1234\\1234\end{vmatrix}\begin{vmatrix}1345\\1234\end{vmatrix}-\begin{vmatrix}1246\\1234\end{vmatrix}\begin{vmatrix}1456\\1234\end{vmatrix}\\
\begin{vmatrix}1235\\1234\end{vmatrix}\begin{vmatrix}2345\\1234\end{vmatrix}-\begin{vmatrix}1256\\1234\end{vmatrix}\begin{vmatrix}2456\\1234\end{vmatrix}\\
\begin{vmatrix}1236\\1234\end{vmatrix}\begin{vmatrix}2346\\1234\end{vmatrix}-\begin{vmatrix}1356\\1234\end{vmatrix}\begin{vmatrix}3456\\1234\end{vmatrix}
\end{Vmatrix}^{*}.$$

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* Muir, ibid.

Similarly the determinant whose elements in the first row are

$$\begin{vmatrix}123\\123\end{vmatrix},\ \begin{vmatrix}145\\123\end{vmatrix}+\begin{vmatrix}246\\123\end{vmatrix}+\begin{vmatrix}356\\123\end{vmatrix},\ \begin{vmatrix}345\\123\end{vmatrix}+\begin{vmatrix}146\\123\end{vmatrix}+\begin{vmatrix}256\\123\end{vmatrix},\ \begin{vmatrix}456\\123\end{vmatrix}$$

$$=\left|\begin{matrix}\begin{vmatrix}1234\\1234\end{vmatrix} & & \begin{vmatrix}3456\\1234\end{vmatrix} & & \begin{vmatrix}2456\\1234\end{vmatrix}\\ \begin{vmatrix}1235\\1234\end{vmatrix} & - & \begin{vmatrix}2456\\1234\end{vmatrix} & - & \begin{vmatrix}1456\\1234\end{vmatrix}\\ \begin{vmatrix}1236\\1234\end{vmatrix} & & \begin{vmatrix}1456\\1234\end{vmatrix} & & \begin{vmatrix}3456\\1234\end{vmatrix}\end{matrix}\right|;$$

and the determinant whose elements in the first row are

$$\begin{vmatrix}123\\123\end{vmatrix},\ \begin{vmatrix}145\\123\end{vmatrix}+\begin{vmatrix}146\\123\end{vmatrix}+\begin{vmatrix}156\\123\end{vmatrix},\ \begin{vmatrix}245\\123\end{vmatrix}+\begin{vmatrix}246\\123\end{vmatrix}+\begin{vmatrix}256\\123\end{vmatrix},\ \begin{vmatrix}456\\123\end{vmatrix}$$

$$=\left|\begin{matrix}\begin{vmatrix}1234\\1234\end{vmatrix} & & \begin{vmatrix}1456\\1234\end{vmatrix} & & \begin{vmatrix}2456\\1234\end{vmatrix}\\ \begin{vmatrix}1235\\1234\end{vmatrix} & - & \begin{vmatrix}1456\\1234\end{vmatrix} & - & \begin{vmatrix}2456\\1234\end{vmatrix}\\ \begin{vmatrix}1236\\1234\end{vmatrix} & & \begin{vmatrix}1456\\1234\end{vmatrix} & & \begin{vmatrix}2456\\1234\end{vmatrix}\end{matrix}\right|=0.$$

SYRACUSE UNIVERSITY,
March 1, 1914.